Semi-infinite linear programs and semi-infinite matrix games

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SEMI-INFINITE LINEAR PROGRAMS AND SEMI-INFINITE MATRIX GAMES

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0. INTRODUCTION

In this paper pairs of semi-infinite programs corresponding to triples $\langle A,b,c\rangle$, where A is an $m\times\infty$-matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^\infty$, are studied by approximating them with finite subprograms.

Sufficient conditions are given to guarantee that there is no duality gap. A subclass of semi-infinite programs is introduced on which the value is a continuous function and on which the solution set of the dual program is an upper semi-continuous multifunction. Also a new proof is presented of the fact that the c-mixed extension of a semi-infinite matrix game has a value and that player I has at least one optimal strategy.

1. SEMI-INFINITE LINEAR PROGRAMS

1.1. Notations

If $m \in \mathbb{N}$, then $\mathbb{N}_m := \{1,2,\ldots,m\}$. For each $a \in \mathbb{R}$: $-\infty+a := -\infty$. $[0,\infty] := \{x \in \mathbb{R};\ x \geq 0\} \cup \{\infty\}$. The zero-element of $\mathbb{R}^m$ is denoted by 0, and $1_m := (1,1,\ldots,1) \in \mathbb{R}^m$. The i-th coordinate of a vector $x \in \mathbb{R}^m$ is denoted by x_i or $(x)_i$. If $x \in \mathbb{R}^m$, then $\|x\|_\infty := \max\{|x_i|;\ i \in \mathbb{N}_m\}$. Let $x,y \in \mathbb{R}^m$. Then $x \geq y$ (or $y \leq x$) if $x_i \geq y_i$ for each $i \in \mathbb{N}_m$, and $x \gg 0$ if $x_i > 0$ for each $i \in \mathbb{N}_m$.

$\mathbb{R}^\infty$ is the linear space of all infinite sequences of real numbers. The zero-element of $\mathbb{R}^\infty$ is denoted by 0, and $1_\infty := (1,1,\ldots) \in \mathbb{R}^\infty$. Let $x,y \in \mathbb{R}^\infty$. Then $x \geq y$ (or $y \leq x$) if $x_i \geq y_i$ for each $i \in \mathbb{N}$.

Let $m \in \mathbb{N} \cup \{\infty\}$ and $x,y \in \mathbb{R}^m$. Then the transpose of y (the column vector corresponding with the row vector y) is denoted by y^t; $xy^t := x_1y_1 + x_2y_2 + \ldots$.

$(\mathbb{R}^\infty)^c$ consists of those elements $q \in \mathbb{R}^\infty$ for which there is an $n \in \mathbb{N}$ such that $q_j = 0$ for each $j > n$. $(\mathbb{R}^n)^c := \mathbb{R}^n$ for $n \in \mathbb{N}$. The set $\{c \in \mathbb{R}^\infty ; \inf_{j\in\mathbb{N}} c_j > 0, \sup_{j\in\mathbb{N}} c_j < \infty\}$ is denoted by $(l^\infty)^{++}$.
Let $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{\infty\}$. An $m\times n$-matrix A with the real number a_{ij} in the (i,j)-th cell $(i \le m, \infty \ne j \le n)$ is also denoted by $[a_{ij}]_{i=1,j=1}^{m\ \ n}$. The set of $m\times n$-matrices is denoted by $M_{m\times n}$. $J := [1]_{i=1,j=1}^{m\ \ n} \in M_{m\times n}$. $L_{m\times n} := \{\langle A,b,c\rangle ; A \in M_{m\times n}; b \in \mathbb{R}^m, c \in \mathbb{R}^n\}$. "W.l.o.g." is used as an abbreviation of "Without loss of generality".

1.2. Let $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{\infty\}$. For $\langle A,b,c\rangle \in L_{m\times n}$ let

$$P(A,b) := \{y \in (\mathbb{R}^n)^c ; y \ge 0, Ay^t \le b^t\},$$

$$D(A,c) := \{x \in \mathbb{R}^m ; x \ge 0, xA \ge c\}.$$

We are interested in the following two problems:

I (*Primal program*). Find

$$v_p(A,b,c) := \sup_{y\in P(A,b)} cy^t$$

and (if possible) an element of

$$O_p(A,b,c) := \{y \in P(A,b) ; cy^t = v_p(A,b,c)\}.$$

II (*Dual program*). Find

$$v_d(A,b,c) := \inf_{x\in D(A,c)} xb^t$$

and (if possible) an element of

$$O_d(A,b,c) := \{x \in D(A,c) ; xb^t = v_d(A,b,c)\}.$$

$P(A,b)$, $v_p(A,b,c)$ and $O_p(A,b,c)$ are respectively called the *feasible region*, the *value* and the *solution set of the primal program* corresponding to $\langle A,b,c\rangle$.

$D(A,c)$, $v_d(A,b,c)$ and $O_d(A,b,c)$ are called the *feasible region*, the *value* and the *solution set of the dual program* corresponding to $\langle A,b,c\rangle$.

1.3. Let $m,n \in \mathbb{N}$ and $\langle A,b,c\rangle \in L_{m\times n}$. It is well-known (see e.g. [8], pp.40-45 or [5]) that for the dual pair of (finite) programs corresponding to <A,b,c> exactly one of the following four assertions holds:

1. The values of both programs are real numbers, both programs have solutions (i.e. $O_p(A,b,c) \neq \emptyset$ and $O_d(A,b,c) \neq \emptyset$) and there is no duality gap (i.e. $v_p(A,b,c) = v_d(A,b,c)$).
2. The primal program is unbounded (i.e. $v_p(A,b,c) = \infty$) and the dual program is infeasible (i.e. $D(A,c) = \emptyset$).
3. The dual program is unbounded (i.e. $v_d(A,b,c) = -\infty$) and the primal program is infeasible (i.e. $P(A,b) = \emptyset$).
4. Both programs are infeasible.

(Note that only in the 4-th case $v_p(A,b,c) \neq v_d(A,b,c)$.)

1.4. Examples

The following examples show that for pairs of semi-infinite programs ($n = \infty$) there are also other possibilities.

1.4.1. Let us give $\langle A,b,c\rangle \in L_{2\times\infty}$ such that the two corresponding programs are feasible and also bounded (i.e. $v_p(A,b,c) \in \mathbb{R}$, $v_d(A,b,c) \in \mathbb{R}$) but where we have a duality gap. Take

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & \dots \end{bmatrix}$$

$b := (1,0)$ and $c := (1,2,2,2,\dots)$. Then

$$P(A,b) = \{y \in \mathbb{R}^{\infty};\ 0 \le y_1 \le 1,\ y_j = 0 \text{ for each } j \in \mathbb{N}\setminus\{1\}\},$$

$$D(A,c) = \{x \in \mathbb{R}^2;\ x_1 \ge 2,\ x_2 \ge 0\}.$$

Hence

$$v_p(A,b,c) = 1 \neq 2 = v_d(A,b,c).$$

1.4.2. To $A := [3/2 \quad 4/3 \quad 5/4 \quad \dots]$, $b := (1)$ and $c := (1,1,\dots)$ there corresponds a pair of bounded programs without a duality gap ($v_p(A,b,c) = v_d(A,b,c) = 1$), where $O_p(A,b,c) = \emptyset$ and $O_d(A,b,c) = \{(1)\} \neq \emptyset$.

1.4.3. Let $b := (0,1)$, $c := (1/2,1/3,1/4,\dots)$ and

$$A := \begin{bmatrix} 1/4 & 1/9 & 1/16 & 1/25 & \dots \\ 1 & 1 & 1 & 1 & \dots \end{bmatrix}.$$

Then $O_p(A,b,c) = P(A,b) = \{(0,0,0,\dots)\} \neq \emptyset$. Further $v_p(A,b,c) = v_d(A,b,c) = 0$ and $O_d(A,b,c) = \emptyset$. (Since $(k^2,1/k) \in D(A,c)$ for each $k \in \mathbb{N}$, we have $v_d(A,b,c) \leq 1/k$. So $v_d(A,b,c) \leq 0$ and it is obvious that $v_d(A,b,c) \geq 0$. For each $x \in D(A,c)$ we have $x_2 > 0$. Hence $O_d(A,b,c) = \emptyset$.)

1.4.4. Now we give a triple $\langle A,b,c\rangle \in L_{1\times\infty}$ for which the primal program is bounded and for which the dual program is infeasible. Let $A := [1 \quad 1/2 \quad 1/3 \quad 1/4 \quad \dots]$, $b := (0)$ and $c := (1,1,1,\dots)$. Then

$$P(A,b) = O_p(A,b,c) = \{(0,0,\dots)\},\ v_p(A,b,c) = 0$$

and

$$D(A,c) = \emptyset,\ v_d(A,b,c) = \infty.$$

1.4.5. Let $b := (-1,1)$, $c := (0,-1,-1,\dots)$ and

$$A := \begin{bmatrix} 0 & -1 & -1 & -1 & \dots \\ 0 & 3/2 & 4/3 & 5/4 & \dots \end{bmatrix}.$$

Then the corresponding primal program is infeasible and

$$D(A,c) = \{x \in \mathbb{R}^2;\ x \geq 0,\ -x_1+x_2 \geq -1\}.$$

So

$$v_d(A,b,c) = -1 \neq -\infty = v_p(A,b,c).$$

1.5. In view of the foregoing examples, it is of interest to find conditions such that no duality gap can occur for pairs of semi-infinite programs. Some such conditions are given in 2.3 and 2.5. For other works in the same field of interest we refer the reader to the papers [1],[2],[3],[4],[6],[7],[9],[10] and [11].

2. APPROXIMATION OF SEMI-INFINITE PROGRAMS WITH FINITE SUBPROGRAMS

2.1. Notations

Let $m \in \mathbb{N}$. If $A = [a_{ij}]_{i=1,j=1}^{m\ \ \infty} \in M_{m\times\infty}$ and $n \in \mathbb{N}$, then $A_n := [a_{ij}]_{i=1,j=1}^{m\ \ n}$. $a_n: \mathbb{R}^n \to \mathbb{R}^\infty$ and $s_n: \mathbb{R}^\infty \to \mathbb{R}^n$ $(n \in \mathbb{N})$ are the maps defined by

$$a_n(x_1,x_2,\dots,x_n) := (x_1,x_2,\dots,x_n,0,0,\dots) \quad \text{for each } x \in \mathbb{R}^n$$

$$s_n(x_1,x_2,\dots) := (x_1,x_2,\dots,x_n) \quad \text{for each } x \in \mathbb{R}^\infty.$$

If $C \subset \mathbb{R}^n$, then $a_n(C) := \{a_n(c);\ c \in C\}$.

2.2. Let $\langle A,b,c\rangle \in L_{m\times\infty}$. We want to compare the corresponding pair of programs with the infinite sequence of finite subprograms associated with the sequence of triples $\langle A_1,b,s_1(c)\rangle$, $\langle A_2,b,s_2(c)\rangle$, ...
We collect some properties in the following

<u>LEMMA 2.2.1.</u>

(2.2.1) $\quad a_n(P(A_n,b)) \subset a_{n+1}(P(A_{n+1},b))$ *for each* $n \in \mathbb{N}$,

$$P(A,b) = \bigcup_{n\in\mathbb{N}} a_n(P(A_n,b).$$

(2.2.2) $\quad D(A_n,s_n(c)) \supset D(A_{n+1},s_{n+1}(c))$ *for each* $n \in \mathbb{N}$,

$$D(A,c) = \bigcap_{n\in\mathbb{N}} D(A_n,s_n(c)).$$

(2.2.3) $\quad v_p(A_n,b,s_n(c)) \le v_p(A_{n+1},b,s_{n+1}(c)) \le v_p(A,b,c)$ *for each* $n \in \mathbb{N}$.

(2.2.4) $\quad v_d(A_n,b,s_n(c)) \le v_d(A_{n+1},b,s_{n+1}(c)) \le v_d(A,b,c)$ *for each* $n \in \mathbb{N}$.

If $P(A_n,b) \neq \emptyset$ *for some* $n \in \mathbb{N}$ *or if* $D(A,c) \neq \emptyset$, *then*

(2.2.5) $\quad v_p(A_k,b,s_k(c)) = v_d(A_k,b,s_k(c))$ *for each* $k \ge n$.

(2.2.6) $\quad \lim_{n\to\infty} v_d(A_n,b,s_n(c)) \le v_d(A,b,c)$.

(2.2.7) $\quad \lim_{n\to\infty} v_p(A_n,b,s_n(c)) = v_p(A,b,c)$.

<u>PROOF</u>. (2.2.1) and (2.2.2) are obvious. (2.2.3) and (2.2.4) follow from (2.2.1) and (2.2.2), respectively. (2.2.5) follows from (2.2.1), (2.2.2) and (the note at the end of) 1.3. (2.2.6) follows from (2.2.4). Let us now prove (2.2.7).

If $P(A,b) = \emptyset$, then in view of (2.2.1) we have

$$\lim_{n\to\infty} v_p(A_n,b,s_n(c)) = \lim_{n\to\infty} (-\infty) = -\infty = v_p(A,b,c).$$

Suppose that $P(A,b) \neq \emptyset$. Note that (2.2.3) implies that

$$\lim_{n\to\infty} v_p(A_n,b,s_n(c)) \leq v_p(A,b,c).$$

Take $r \in \mathbb{R}$ such that $r < v_p(A,b,c)$. Then there exists a $y \in P(A,b)$ such that $cy^t > r$. In view of (2.2.1) there is a $k \in \mathbb{N}$ and a $z \in P(A_k,b)$ such that $a_k(z) = y$. Then for each $n \geq k$ we have

$$v_p(A_n,b,s_n(c)) \geq v_p(A_k,b,s_k(c)) \geq s_k(c)z^t = cy^t > r.$$

So we may conclude that $\lim_{n\to\infty} v_p(A_n,b,s_n(c)) \geq v_p(A,b,c)$. Hence (2.2.7) holds. □

THEOREM 2.3. *Let* $\langle A,b,c\rangle \in L_{m\times\infty}$ *and suppose that* $P(A,b) \neq \emptyset$ *or* $D(A,c) \neq \emptyset$. *Then the two following assertions are equivalent:*

(1) $v_p(A,b,c) = v_d(A,b,c)$;

(2) $\lim_{n\to\infty} v_d(A_n,b,s_n(c)) = v_d(A,b,c)$.

PROOF. If $P(A,b) \neq \emptyset$, then it follows from (2.2.1) that there exists an $n \in \mathbb{N}$ such that $P(A_n,b) \neq \emptyset$. Then

$$v_p(A_k,b,s_k(c)) = v_d(A_k,b,s_k(c)) \qquad \text{for each } k \geq n$$

in view of (2.2.5). With the aid of (2.2.7) we may conclude that (1) and (2) are equivalent. If $D(A,c) \neq \emptyset$, then $D(A_n,s_n(c)) \neq \emptyset$ for each $n \in \mathbb{N}$. And then it follows analogously that (1) and (2) are equivalent. □

2.4. REMARKS.

1. In the examples 1.4.1, 1.4.4 and 1.4.5 we have respectively

$$\lim_{n\to\infty} v_d(A_n,b,s_n(c)) = \lim_{n\to\infty} 1 = 1 < 2 = v_d(A,b,c),$$

$$\lim_{n\to\infty} v_d(A_n,b,s_n(c)) = \lim_{n\to\infty} 0 = 0 < \infty = v_d(A,b,c),$$

$$\lim_{n\to\infty} v_d(A_n,b,s_n(c)) = \lim_{n\to\infty} (-\infty) = -\infty < -1 = v_d(A,b,c).$$

2. It is not difficult to prove that

$$\lim_{n\to\infty} v_d(A_n,b,s_n(c)) = v_d(A,b,c)$$

for each $\langle A,b,c\rangle \in L_{1\times\infty}$ with $D(A,c) \neq \emptyset$. So in some sense 1.4.5 is an example of smallest size where the primal program is infeasible and the dual program is bounded.

2.5. The following theorem asserts that a pair of programs corresponding to a triple $\langle A,b,c\rangle \in L_{m\times\infty}$ with $b >> 0$ has no duality gap.

<u>THEOREM</u> (*Duality*). *Let* $\langle A,b,c\rangle \in L_{m\times\infty}$ *and suppose that* $b >> 0$. *Either*
(1) $D(A,c) = \emptyset$ *and* $v_p(A,b,c) = \infty$ $(= v_d(A,b,c))$, *or*
(2) $v_p(A,b,c) = v_d(A,b,c) \in [0,\infty)$ *and* $O_d(A,b,c) \neq \emptyset$.

<u>PROOF</u>. $b >> 0$ implies that $(0,0,\ldots,0) \in P(A_n,b) \neq \emptyset$ for each $n \in \mathbb{N}$. Hence, in view of (2.2.5), we have

$$(*) \qquad v_p(A_n,b,s_n(c)) = v_d(A_n,b,s_n(c)) \in [0,\infty] \quad \text{for each } n \in \mathbb{N}.$$

(a) First suppose that $\lim_{n\to\infty} v_p(A_n,b,s_n(c)) = \infty$. Then (2.2.6), (2.2.7) and (*) imply that

$$\infty = v_p(A,b,c) \leq v_d(A,b,c)$$

and thus, $D(A,c) = \emptyset$. So we are in case (1).

(b) Now suppose that $v := \lim_{n\to\infty} v_p(A_n,b,s_n(c)) \in [0,\infty)$. Then

$$0 \leq v_d(A_n,b_n,s_n(c)) = v_p(A_n,b_n,s_n(c)) < \infty \qquad \text{for each } n \in \mathbb{N}$$

implies, by virtue of 1.3, that $O_d(A_n,b,s_n(c)) \neq \emptyset$ for each $n \in \mathbb{N}$. Take for each $n \in \mathbb{N}$ an $x^n \in O_d(A_n,b,s_n(c))$. Then the sequence $\langle x^n\rangle$ lies in the set

$$\{x \in \mathbb{R}^m;\ x \geq 0,\ xb^t \leq v\}$$

and this set is compact because $b >> 0$. So we may suppose w.l.o.g.

that $<x^n>$ converges to an element $\hat{x} \in \mathbb{R}^m$. Then it is obvious that

$$\hat{x} \in \bigcap_{n \in \mathbb{N}} D(A_n, s_n(c)) = D(A,c) \neq \emptyset.$$

(This follows from (2.2.2) and the fact that $D(A_n, s_n(c))$ is a closed subset of $\mathbb{R}^m$ for each $n \in \mathbb{N}$.) From

$$\hat{x}b^t = \lim_{n\to\infty} x^n b^t = \lim_{n\to\infty} v_d(A_n, b, s_n(c)) = v$$

it follows that $v_d(A,b,c) \leq \hat{x}b^t = v$.

On the other hand, we have $v_d(A,b,c) \geq v$ because of (2.2.6) and (*). Hence $v_d(A,b,c) = v$, $\hat{x} \in O_d(A,b,c) \neq \emptyset$. Moreover $v_p(A,b,c) = v$ in view of (2.2.7). Thus we are in case (2).

This completes the proof. □

2.6. <u>EXAMPLES</u>.

1. Let $A := [1 \;\; 1/2 \;\; 1/3 \;\; \ldots]$, $b := (1) \gg 0$ and $c := (1,1,\ldots)$. Then $D(A,c) = \emptyset$ and $v_p(A,b,c) = \infty$ (Case (1) of Theorem 2.5).
2. Let $A := [1 \;\; 1 \;\; 1 \;\; \ldots]$, $b := (1) \gg 0$ and $c := (1,1,\ldots)$. Then $O_d(A,b,c) = \{(1)\}$ and $v_p(A,b,c) = v_d(A,b,c) = 1$ (Case (2) of Theorem 2.5).

2.7. Now we want to consider the following problems:
1. For which $<A,b,c> \in L_{m\times\infty}$ with $v_p(A,b,c) \in \mathbb{R}$ is $O_p(A,b,c) \neq \emptyset$?
2. What is the connection between the solution sets of semi-infinite programs and those of the corresponding finite subprograms?

Therefore we define for elements $<A,b,c> \in L_{m\times\infty}$ the *critical number*, $cr(A,b,c)$, as follows:

$$cr(A,b,c) := \infty \quad \text{if } v_p(A_n,b,s_n(c)) < v_p(A,b,c) \text{ for each } n \in \mathbb{N}$$

and

$$cr(A,b,c) := \min\{n \in \mathbb{N};\; v_p(A_n,b,s_n(c)) = v_p(A,b,c)\} \quad \text{otherwise.}$$

<u>2.8. THEOREM</u> (cf. [12], theorem II.3.9). *Let* $<A,b,c> \in L_{m\times\infty}$ *and suppose that* $v_p(A,b,c) \in \mathbb{R}$. *Then we have*

(1) $cr(A,b,c) < \infty$ *iff* $O_p(A,b,c) \neq \emptyset$;

(2) *if* $cr(A,b,c) < \infty$, *then* $O_p(A,b,c) = \bigcup_{n \geq cr(A,b,c)} a_n(O_p(A_n,b,s_n(c)))$;

(3) *if* $cr(A,b,c) < \infty$ *and* $b >> 0$,

then $O_d(A,b,c) = \bigcap_{n \geq cr(A,b,c)} O_d(A_n,b,s_n(c))$.

PROOF. (a) First suppose that $cr(A,b,c) = \infty$. For each $y \in P(A,b)$ there is an $n \in \mathbb{N}$ such that $y_j = 0$ for each $j > n$. Then $s_n(y) \in P(A_n,b)$ and

$$cy^t = s_n(c)(s_n(y))^t \leq v_p(A_n,b,s_n(c)) < v_p(A,b,c).$$

So $y \notin O_p(A,b,c)$; $O_p(A,b,c) = \emptyset$.

Now suppose that $cr(A,b,c) < \infty$. Then $P(A_n,b) \neq \emptyset$ for each $n \geq$ $\geq cr(A,b,c)$. In view of 1.3 we may conclude that $O_p(A_n,b,s_n(c)) \neq \emptyset$ for each $n \geq cr(A,b,c)$. For $n \geq cr(A,b,c)$, let $y \in O_p(A_n,b,s_n(c))$. Then $a_n(y) \in P(A,b)$ and

$$c(a_n(y))^t = s_n(c)y^t = v_p(A_n,b,s_n(c)) = v_p(A,b,c).$$

So $a_n(y) \in O_p(A,b,c) \neq \emptyset$ and we have proved (1). Moreover we have shown that

$$O_p(A,b,c) \supset \bigcup_{n \geq cr(A,b,c)} a_n(O_p(A_n,b,s_n(c))).$$

Let $cr(A,b,c) < \infty$ and $y \in O_p(A,b,c)$. Take an $n \geq cr(A,b,c)$ such that $y_j = 0$ for each $j > n$. Then

$$s_n(c)s_n(y)^t = cy^t = v_p(A,b,c) = v_p(A_n,b,s_n(c)).$$

So $s_n(y) \in O_p(A_n,b,s_n(c))$ and we have proved that

$$O_p(A,b,c) \subset \bigcup_{n \geq cr(A,b,c)} a_n(O_p(A_n,b,s_n(c))).$$

This completes the proof of (2).

(b) Now suppose that $cr(A,b,c) < \infty$ and that $b >> 0$. Since $v_p(A,b,c) \in$ $\in \mathbb{R}$ we may conclude from 2.5 that

$$v_p(A,b,c) = v_d(A,b,c) \quad \text{and} \quad O_d(A,b,c) \neq \emptyset.$$

Take an $x \in O_d(A,b,c)$. Then, for each $n \geq cr(A,b,c)$ we have

$$xb^t = v_d(A,b,c) = v_p(A,b,c) = v_p(A_n,b,s_n(c)) =$$

$$= v_d(A_n,b,s_n(c)).$$

So

$$x \in \bigcap_{n \geq cr(A,b,c)} O_d(A_n,b,s_n(c)).$$

Now take an $\bar{x} \in \bigcap_{n \geq cr(A,b,c)} O_d(A_n,b,s_n(c))$. Then $x \in D(A,c)$ and for each $n \geq cr(A,b,c)$:

$$xb^t = v_d(A_n,b,s_n(c)) = v_p(A_n,b,s_n(c)) =$$

$$= v_p(A,b,c) = v_d(A,b,c).$$

So $x \in O_d(A,b,c)$.

This completes the proof of (3). □

2.9. EXAMPLE.

Let $A := [2 \quad 3/2 \quad 4/3 \quad 5/4 \quad \ldots]$, $b := (1) \gg 0$ and $c := (1,1,1,\ldots)$. Then $v_p(A,b,c) = v_d(A,b,c) = 1$, $O_p(A,b,c) = \emptyset$ and $O_d(A,b,c) = \{(1)\}$. Note that $v_p(A_n,b,s_n(c)) = n/n+1 < 1 = v_p(A,b,c)$; $cr(A,b,c) = \infty$.

3. A SUBCLASS OF SEMI-INFINITE LINEAR PROGRAMS

3.1. Let $m \in \mathbb{N}$ and let $\bar{d}\colon M_{m\times\infty} \times M_{m\times\infty} \to [0,\infty]$ be the metric defined by

$$\bar{d}(A,B) := \sup\{|a_{ij}-b_{ij}|;\ i \in \mathbb{N}_m,\ j \in \mathbb{N}\}$$

for each $A = [a_{ij}]_{i=1}^{m}{}_{,j=1}^{\infty}$, $B = [b_{ij}]_{i=1}^{m}{}_{,j=1}^{\infty} \in M_{m\times\infty}$. Let $d\colon L_{m\times\infty} \times L_{m\times\infty} \to [0,\infty]$ be the metric on $L_{m\times\infty}$ defined by

$$d(\langle A,b,c\rangle,\langle A',b',c'\rangle) := \max\{\bar{d}(A,A'),\|b-b'\|_\infty,\|c-c'\|_\infty\}$$

for each $\langle A,b,c\rangle,\langle A',b',c'\rangle \in L_{m\times\infty}$. ($\|c-c'\|_\infty := \sup_{j\in\mathbb{N}} |c_j-c'_j| \in [0,\infty]$.)

We endow $L_{m\times\infty}$ with the topology induced by the metric d.

3.2. Let

$$F_{m\times\infty} := \{A \in M_{m\times\infty};\ D(A,1_\infty) \neq \emptyset\}$$

and

$$P_{m\times\infty} := \{\langle A,b,c\rangle \in L_{m\times\infty};\ A \in F_{m\times\infty};\ b \gg 0,\ c \in (l^\infty)^{++}\}.$$

In this section we study semi-infinite programs corresponding to elements of $P_{m\times\infty}$.

<u>LEMMA</u>. $P_{m\times\infty}$ *is an open subset of* $L_{m\times\infty}$.

<u>PROOF</u>. It is sufficient to show that $F_{m\times\infty}$ is an open subset of $M_{m\times\infty}$. Take $A \in F_{m\times\infty}$ and let $x \in D(A,1_\infty)$. Then for each $B \in M_{m\times\infty}$ with $\bar{d}(A,B) < (\sum_{i=1}^m x_i)^{-1}$ we have

$$xB = xA+x(B-A) \geq (1 - \sum_{i=1}^{m} x_i\ \bar{d}(A,B))1_\infty,$$

$$(1 - \sum_{i=1}^{m} x_i\ \bar{d}(A,B))^{-1}\ x \in D(B,1_\infty).$$

So $B \in F_{m\times\infty}$, and the lemma is proved. □

3.3. <u>THEOREM</u>. *Let* $\langle A,b,c\rangle \in P_{m\times\infty}$. *Then*

$$(*) \qquad v_p(A,b,c) = v_d(A,b,c) \in (0,\infty) \quad \textit{and} \quad O_d(A,b,c) \neq \emptyset.$$

<u>PROOF</u>. Let $\tau = \sup_{j\in\mathbb{N}} c_j < \infty$ and let $x \in D(A,1_\infty)$. Then

$$(\tau x)A \geq \tau 1_\infty \geq c, \quad \tau x \in D(A,c) \neq \emptyset.$$

So we may conclude that (*) holds by virtue of 2.5. □

3.4. <u>THEOREM</u>. *Let* $m \in \mathbb{N}$. *Then*

$v_d\colon P_{m\times\infty} \to \mathbb{R}$ *is a continuous function,*

$O_d\colon P_{m\times\infty} \to \mathbb{R}^m$ *is an upper semi-continuous multifunction.*

<u>PROOF</u>. Given $\langle A,b,c\rangle \in P_{m\times\infty}$ and given an infinite sequence $\langle A^1,b^1,c^1\rangle, \langle A^2,b^2,c^2\rangle, \ldots$ in $P_{m\times\infty}$ such that

$$\alpha^n := \bar{d}(A^n,A) \to 0,$$
$$\beta^n := \|b^n-b\|_\infty \to 0, \qquad \text{if } n \to \infty$$
$$\gamma^n := \|c^n-c\| \to 0$$

we have to show that

$$\lim_{n\to\infty} v_d(A^n,b^n,c^n) = v_d(A,b,c).$$

(a) First we prove that

(*) $$\limsup_{n\to\infty} v_d(A^n,b^n,c^n) \le v_d(A,b,c).$$

Take an $x \in O_d(A,b,c)$ and put $\xi := \sum_{i=1}^m x_i$, $\sigma := \inf_{j\in\mathbb{N}} c_j > 0$ and $\sigma^n := \inf_{j\in\mathbb{N}} (c^n)_j > 0$ for each $n \in \mathbb{N}$ (note that $\lim_{n\to\infty} \sigma^n = \sigma$). For each $n \in \mathbb{N}$ we have

(1) $$xA^n = xA+x(A^n-A \ge c-\xi\alpha^n 1_\infty.$$

Take an $N \in \mathbb{N}$ such that $\sigma-\xi\alpha^n > 0$ for each $n \ge N$. It follows from (1) that

(2) $$(\sigma-\xi\alpha^n)^{-1} xA^n \ge 1_\infty \quad \text{for each } n \ge N.$$

Let $y^n := (1 + (\xi\alpha^n+\gamma^n)(\sigma-\xi\alpha^n)^{-1})x$ for each $n \ge N$. Then (2) and (1) imply that for each $n \ge N$ we have

$$y^nA^n \ge xA^n + (\xi\alpha^n+\gamma^n)1_\infty \ge c-\xi\alpha^n 1_\infty + (\xi\alpha^n+\gamma^n)1_\infty \ge c^n.$$

So $y^n \in D(A^n,c^n)$ and

(3) $$v_d(A^n,b^n,c^n) \le y^nb^n \quad \text{for } n \ge N.$$

Since $\lim_{n\to\infty} y^n = x$ we have $\lim_{n\to\infty} y^nb^n = v_d(A,b,c)$; hence formula (3) implies (*).

(b) Now we prove that

(**) $$\liminf_{n\to\infty} v_d(A^n,b^n,c^n) \ge v_d(A,b,c).$$

Suppose that (**) is not true. Then there exists a subsequence

$$\langle A^{n(1)}, b^{n(1)}, c^{n(1)}\rangle, \langle A^{n(2)}, b^{n(2)}, c^{n(2)}\rangle, \ldots$$

of the sequence $\langle A^1, b^1, c^1\rangle, \langle A^2, b^2, c^2\rangle, \ldots$ such that

$$p := \lim_{k\to\infty} v_d(A^{n(k)}, b^{n(k)}, c^{n(k)}) < v_d(A,b,c).$$

For each $k \in \mathbb{N}$ take an $x^{n(k)} \in O_d(A^{n(k)}, b^{n(k)}, c^{n(k)})$; and put

$$\rho := \min_{i\in\mathbb{N}_m} b_i > 0,$$

$$\rho^{n(k)} := \min_{i\in\mathbb{N}_m} (b^{n(k)})_i > 0 \quad \text{for } k \in \mathbb{N}.$$

Then

$$\rho^{n(k)} \sum_{i=1}^{m} (x^{n(k)})_i \leq x^{n(k)} (b^{n(k)})^t = v_d(A^{n(k)}, b^{n(k)}, c^{n(k)}).$$

From

$$\sum_{i=1}^{m} (x^{n(k)})_i \leq (\rho^{n(k)})^{-1} v_d(A^{n(k)}, b^{n(k)}, c^{n(k)}) \to \rho^{-1} p \quad \text{if } k \to \infty$$

we may conclude that $x^{n(1)}, x^{n(2)}, \ldots$ is a bounded sequence in $\mathbb{R}^m$. W.l.o.g. we suppose that this sequence converges to an element $\hat{x} \in \mathbb{R}^m$. It follows (by taking limits) from $x^{n(k)}A^{n(k)} \geq c^{n(k)}$ and the fact that $\lim_{k\to\infty} x^{n(k)}b^{n(k)} = p$, that $\hat{x} \in D(A,c)$ and $\hat{x}b^t = p < v_d(A,b,c)$. But that is a contradiction. So (**) is proved.

From (*) and (**) it follows that $v_d\colon P_{m\times\infty} \to \mathbb{R}$ is a continuous function.

(c) Suppose that $O_d\colon P_{m\times\infty} \to \mathbb{R}^m$ is not an upper semi-continuous multi-function. Then there exist a triple $\langle A,b,c\rangle \in P_{m\times\infty}$, an open subset U of $\mathbb{R}^m$ with $U \supset O_d(A,b,c)$, and a sequence $\langle A^1, b^1, c^1\rangle, \langle A^2, b^2, c^2\rangle, \ldots$ in $P_{m\times\infty}$ such that for each $n \in \mathbb{N}$ there is an $x^n \in O_d(A^n, b^n, c^n)$ with $x^n \notin U$ and such that

$$\bar{d}(A,A^n) < n^{-1}, \quad \|b-b^n\|_\infty < n^{-1} \quad \text{and} \quad \|c-c^n\|_\infty < n^{-1}.$$

In a similar way as in part (b) of this proof, it follows that $x^1, x^2, \ldots$ is a bounded sequence in $\mathbb{R}^m$ and that for a limit point, $\hat{x}$, of that sequence we have $\hat{x} \notin U$ and $\hat{x} \in O_d(A,b,c) \subset U$. But this is impossible. Hence $O_d\colon P_{m\times\infty} \to \mathbb{R}^m$ is an upper semi-continuous multifunction. □

3.5. EXAMPLE. Let $A := [1 \;\; 1/2 \;\; 1/3 \;\ldots] \in M_{1\times\infty}$, $b := (1) \in \mathbb{R}^1$ and $c := (1,1/2,1/3,\ldots) \in \mathbb{R}^\infty$. Then $v_{P_\infty}(A,b,c) = v_d(A,b,c) = 1$ and $O_d(A,b,c) = \{(1)\}$. Note that $c \notin (1^\infty)^{++}$ and thus $\langle A,b,c\rangle \notin P_{m\times\infty}$. For each $n \in \mathbb{N}$ let

$$A(n) := [1 \;\; 1/2 \;\ldots\; 1/n \;\; 0 \;\; 1/(n+2) \;\ldots] \in M_{1\times\infty}$$

$$c(n) := (1, 1/2,\ldots,1/(n-1), 2/n, 1/(n+1), 1/(n+2),\ldots) \in \mathbb{R}^\infty.$$

Then $d(\langle A,b,c\rangle, \langle A(n),b,c\rangle) = 1/n+1 \to 0$ if $n \to \infty$ and $D(A(n),c) = \emptyset$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} v_p(A(n),b,c) = \lim \infty = \infty \neq v_p(A,b,c)$. Also $d(\langle A,b,c\rangle, \langle A,b,c(n)\rangle) = 1/n \to 0$ if $n \to \infty$; and $\lim_{n\to\infty} v_d(A,b,c(n)) = \lim_{n\to\infty} 2 = 2 \neq 1 = v_d(A,b,c)$, and for each $n \in \mathbb{N}$: $O_d(A,b,c(n)) = \{2\} \not\subset U$ if U is the open set $(0,2)$ in $\mathbb{R}^1$ containing $\{1\} = O_d(A,b,c)$.

4. SEMI-INFINITE MATRIX GAMES

4.1. Notations

Let $m \in \mathbb{N}$. Then $S^m := \{p \in \mathbb{R}^m;\; p \geq 0,\; \sum_{i=1}^m p_i = 1\}$. $S^c := \{x \in (\mathbb{R})^c;\; x \geq 0,\; \sum_{i=1}^\infty x_i = 1\}$.

4.2. DEFINITIONS. Let $A = [a_{ij}]_{i=1,j=1}^{m\;\;\infty} \in M_{m\times\infty}$ $(m \in \mathbb{N})$. The zero-sum two-person game, $\langle S^m, S^c, E_A\rangle$ with

$$E_A(p,q) := pAq^t = \sum_{i=1}^m \sum_{j=1}^\infty p_i a_{ij} q_j \quad \text{for each } (p,q) \in S^m \times S^c$$

is called the *c-mixed extension of the matrix game* A. (S^m is called the set of *mixed strategies* of player I in the matrix game A and S^c the set of *c-mixed strategies* of player II. $E_A\colon S^m\times S^c \to \mathbb{R}$ is called the *pay off function* (of player I) in the c-mixed extension.)

The *lower value* $\sup_{p\in S^m} \inf_{q\in S^c} pAq^t$ of the game $\langle S^m,S^c,E_A\rangle$ is denoted by $\underline{v}(A)$ and the *upper value* $\inf_{q\in S^c} \sup_{p\in S^m} pAq^t$ by $\bar{v}(A)$. $O_I(A) := \{\hat{p} \in S^m;\; \inf_{q\in S^c} \hat{p}Aq^t = \underline{v}(A)\}$.

4.3. LEMMA. *Let* $A,B \in M_{m\times\infty}$, $\varepsilon > 0$, $c \in \mathbb{R}$ *and* $J := [1]_{i=1,j=1}^{m\ \ \infty} \in M_{m\times\infty}$. *Then*

(4.3.1) $\quad -\infty \le \underline{v}(A) \le \bar{v}(A) < \infty.$

(4.3.2) $\quad \underline{v}(A+cJ) = \underline{v}(A)+c, \quad \bar{v}(A+cJ) = \bar{v}(A)+c, \quad O_I(A+cJ) = O_I(A).$

(4.3.3) $\quad$ *If* $\|B-A\|_\infty \le \varepsilon$, *then* $\underline{v}(A)-\varepsilon \le \underline{v}(B) \le \underline{v}(A)+\varepsilon.$

(4.3.4) $\quad \underline{v}(A) > 0$ *iff* $D(A,1_\infty) \ne 0.$

PROOF. We only prove (4.3.4) and leave the proofs of the first three assertions to the reader.

First suppose that $\underline{v}(A) > 0$. Since

$$p \mapsto \inf_{q\in S^c} pAq^t \qquad (p \in S^m)$$

is an upper semi-continuous function on the compact set S^m, there exists a $\hat{p} \in S^m$ such that

$$\inf_{q\in S^c} \hat{p}Aq^t = \max_{p\in S^m} \inf_{q\in S^c} pAq^t = \underline{v}(A).$$

Hence $\hat{p} \in O_I(A)$; $\underline{v}(A)^{-1}\hat{p}A \ge 1_\infty$ and thus $\underline{v}(A)^{-1}\hat{p} \in D(A,1_\infty) \ne \emptyset$. So we have proved the implication to the right in (4.3.4).

Now suppose that $x \in D(A,1_\infty)$. Then $\hat{x} := (\sum_{i=1}^m x_i)^{-1}x \in S^m$ and

$$\underline{v}(A) \ge \inf_{q\in S^c} \hat{x}Aq^t \ge \Big(\sum_{i=1}^m x_i\Big)^{-1} > 0.$$

Thus the implication to the left in (4.3.4) is also proved. □

4.4. LEMMA. *Let* $A \in M_{m\times\infty}$ *and suppose that* $\underline{v}(A) > 0$. *Then*

(i) $\langle A,1_m,1_\infty\rangle \in P_{m\times\infty}$;

(ii) $\underline{v}(A) = \bar{v}(A) = (v_d(A,1_m,1_\infty))^{-1}$;

(iii) $O_I(A) = \{\underline{v}(A)x;\ x \in O_d(A,1_m,1_\infty)\} \ne \emptyset$.

PROOF. $\langle A,1_m,1_\infty\rangle \in P_{m\times\infty}$ because $1_m \gg 0$ and $1_\infty \in (l^\infty)^{++}$ and because $D(A,1_\infty) \ne \emptyset$ in view of (4.3.4). Thus (i) is proved.

Let $\hat{p} \in O_I(A)$. Then $\underline{v}(A)^{-1}\hat{p} \in D(A,1_\infty)$ (see the proof of 4.3). Hence

(1) $$O_I(A) \subset \{\underline{v}(A)x;\ x \in D(A,1_\infty),\ x1_m^t = (\underline{v}(A))^{-1}\},$$

and

(2) $$v_d(A,1_m,1_\infty) \le (\underline{v}(A))^{-1}\hat{p}1_m^t = (\underline{v}(A))^{-1}.$$

Let $\varepsilon \in (0,v_p(A,1_m,1_\infty))$. Take an $x \in O_d(A,1_m,1_\infty)$ and take an $y^\varepsilon \in P(A,1_m)$ such that $\eta := 1_\infty(y^\varepsilon)^t \ge v_p(A,1_m,1_\infty)-\varepsilon$. (In view of (i) and 3.3 we have: $O_d(A,1_m,1_\infty) \ne \emptyset$, $v_p(A,1_m,1_\infty) \in (0,\infty)$ and so $P(A,1_m) \ne \emptyset$.) Let $\hat{x} := (v_d(A,1_m,1_\infty))^{-1} x \in S^m$, $\hat{y}^\varepsilon := \eta^{-1}y^\varepsilon \in S^c$. Then

(3) $$\underline{v}(A) \ge \inf_{q\in S^c} \hat{x}Aq^t \ge v_d(A,1_m,1_\infty))^{-1}$$

and

(4) $$\bar{v}(A) \le \sup_{p\in S^m} pA(\hat{y}^\varepsilon)^t \le \eta^{-1} \le (v_p(A,1_m,1_\infty)-\varepsilon)^{-1}$$

for each $\varepsilon > 0$.

By (i) and 3.3 we have: $v_p(A,1_m,1_\infty) = v_d(A,1_m,1_\infty)$. So we may conclude from (3), (4) and (4.3.1) that

(5) $$\underline{v}(A) = \bar{v}(A) = (v_d(A,1_m,1_\infty))^{-1}$$

and we have proved (ii).

Now (3) and (5) imply that

(6) $$\underline{v}(A)x \in O_I(A) \quad \text{for each } x \in O_d(A,1_m,1_\infty)$$

and (1) and (5) imply that

(7) $$O_I(A) \subset \{\underline{v}(A)x;\ x \in O_d(A,1_m,1_\infty)\}.$$

But then (iii) follows from (6) and (7). □

4.5. (*Minimax*) <u>THEOREM</u>. *Let* $A \in M_{m\times\infty}$. *Then* $\underline{v}(A) = \bar{v}(A)$ *and* $O_I(A) \ne \emptyset$.

<u>PROOF</u>. (a) First suppose that $\underline{v}(A) \in \mathbb{R}$. Then we can take $c \in \mathbb{R}$ such that $\underline{v}(A+cJ) = \underline{v}(A)+c > 0$ (see (4.3.2)). By 4.4 we may conclude that $\underline{v}(A+cJ) = \bar{v}(A+cJ)$ and that $O_I(A+cJ) \ne \emptyset$. But then $\underline{v}(A) = \bar{v}(A)$ and $O_I(A) \ne \emptyset$ in view of (4.3.2).

(b) Now suppose that $\underline{v}(A) = -\infty$. Then $\underline{v}(A+cJ) = -\infty$ and thus $D(A+cJ,1_\infty) = \emptyset$ for each $c \in \mathbb{R}$ (see (4.3.2) and (4.3.4)). It follows from Theorem 2.5 that $v_p(A+cJ,1_m,1_\infty) = \infty$ for each $c \in \mathbb{R}$.

To prove that $\bar{v}(A) = -\infty$ we show that $\bar{v}(A) \leq 1-c$ for each $c \in \mathbb{R}$. Take $c \in \mathbb{R}$ and $q \in P(A+cJ,1_m)$ such that $s := \sum_{j\in\mathbb{N}} q_j = 1_\infty q^t \geq 1$. Then $\hat{q} = s^{-1}q \in S^c$, $(A+cJ)\hat{q}^t \leq s^{-1}1_m$ and so

$$\bar{v}(A+cJ) \leq \sup_{p\in S^m} p(A+cJ)\hat{q}^t \leq \sup_{p\in S^m} p(s^{-1}1_m^t) = s^{-1} \leq 1.$$

Then $\bar{v}(A) = \bar{v}(A+cJ)-c \leq 1-c$. Hence $\bar{v}(A) = -\infty = \underline{v}(A)$. Further $O_I(A) = S^m \neq \emptyset$. $\square$

4.6. REMARK. Theorem 4.5 is well-known; the proof given here is new. For other proofs see A.L. SOYSTER [11], and S.H. TIJS [12], theorem II.2.2 and also remark II.2.5.

4.7. THEOREM. *Let* $m \in \mathbb{N}$. *Then*

$\underline{v}: M_{m\times\infty} \to \mathbb{R}$ *is a continuous function.*

$O_I: M_{m\times\infty} \to S^m$ *is an upper semi-continuous multifunction.*

PROOF. The continuity of $\underline{v}: M_{m\times\infty} \to \mathbb{R}$ follows immediately from (4.3.3). To prove that $O_I: M_{m\times\infty} \to S^m$ is upper semi-continuous we have to show that for each $A \in M_{m\times\infty}$ and for each open set U in $\mathbb{R}^m$ with $U \supset O_I(A)$ there exists a $\delta > 0$ such that $O_I(B) \subset U$ for each $B \in M_{m\times\infty}$ with $\|B-A\|_\infty < \delta$. We consider three cases.

(a) Let $A \in M_{m\times\infty}$ and suppose that $\underline{v}(A) > 0$. Take an open set U in $\mathbb{R}^m$ with $U \supset O_I(A)$. Let $U^* := \{tx;\ t > 0,\ x \in U\}$. Then U^* is an open set in $\mathbb{R}^m$, $U^* \cap S^m = U \cap S^m$, and $O_d(A,1_m,1_\infty) \subset U^*$ in view of 4.4 (iii). It follows from 3.4 that there is a $\delta \in (0,\underline{v}(A))$ such that $O_d(B,1_m,1_\infty) \subset U^*$ for each $B \in M_{m\times\infty}$ with $\|B-A\|_\infty < \delta$. Then (4.3.3) and 4.4 imply that $O_I(B) \subset U^* \cap S^m = U \cap S^m \subset U$ for each $B \in M_{m\times\infty}$ with $\|B-A\|_\infty < \delta$.

(b) Let $A \in M_{m\times\infty}$ such that $\underline{v}(A) \in (-\infty,0]$. Let U be open in $\mathbb{R}^m$ and $U \supset O_I(A)$. Take $c \in \mathbb{R}$ such that $\underline{v}(A+cJ) > 0$. Then $U \supset O_I(A+cJ) = O_I(A)$. In view of (a) we can find a $\delta > 0$ such that for each $B \in M_{m\times\infty}$ with $\|B-A\|_\infty < \delta$ we have $O_I(B) = O_I(B+cJ) \subset U$.

(c) Let $A \in M_{m\times\infty}$ and $\underline{v}(A) = -\infty$. Then for each $B \in M_{m\times\infty}$ we have

$$O_I(B) \subset O_I(A) = S^m. \qquad \square$$

4.8. REMARKS.

1. Another proof of theorem 4.7 was given in [12] (theorems II.7.2 and II.7.5).
2. Theorem 3.4 can also be derived from theorem 4.7 using the following facts: Let $\langle A,b,c\rangle \in P_{m\times\infty}$, where $A = [a_{ij}]_{i=1,j=1}^{m\ \ \infty}$, and let $\hat{A} := [a_{ij}/b_i c_j]_{i=1,j=1}^{m\ \ \infty}$. Then
   (1) $v_d(A,b,c) = v_d(\hat{A},1_m,1_\infty) = v(\hat{A})^{-1}$,
   (2) $O_I(\hat{A}) = \{\underline{v}(\hat{A})x;\ x \in O_d(\hat{A},1_m,1_\infty)\}$,
   (3) $O_d(\hat{A},1_m,1_\infty) = \{(b_1x_1,b_2x_2,\ldots,b_mx_m);(x_1,\ldots,x_m) \in O_d(A,b,c)\}$.

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